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1984 J. Phys. A: Math. Gen. 17 3201

(http://iopscience.iop.org/0305-4470/17/16/021)

J. Phys. A: Math Gen. 17 (1984) 3201-3207. Printed in Great Britain

Eight-vertex model on a ruby lattice

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Received 21 February 1984

Abstract. An eight-vertex model on a ruby lattice with staggered (site-dependent) vertex weights is considered. The special case of a free-fermion model is solved by the Pfaffian method. In general the specific heat has logarithmic singularity at the critical temperature, except in some special cases where the system exhibits a first- or second-order phase transition.

1. Introduction

Recently the residual entropy of two-dimensional ice (ice model) on a ruby lattice (figure 1) was calculated by the method of series expansion (Lin and Ma 1983a) and the Ising model on a ruby lattice was solved exactly by the method of Pfaffian (Lin and Ma 1983b). The ice model is a special case of the eight-vertex model. The eight-vertex model on a square lattice was solved exactly by Baxter (1971).

The staggered eight-vertex model on a square lattice which allows different vertex weights for the two sublattices was studied by Hsue *et al* (1975), they used the Pfaffian method to solve exactly the special case where the vertex weights satisfy the so-called free-fermion conditions (Fan and Wu 1970). Their result was generalised to four sublattices by Lin and Wang (1977). The Pfaffian solution of the eight-vertex model on a Kagomé lattice was derived by Lin (1976). The purpose of this paper is to study the Pfaffian solution of the eight-vertex model on a ruby lattice.

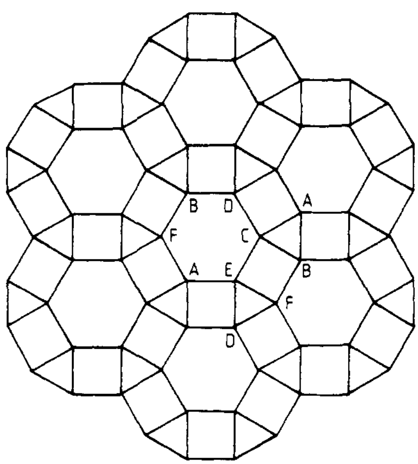

Figure 1. A ruby lattice with six sublattices A, B, C, D, E and F.

2. Definition of the model

Arrows are placed on the bonds of a ruby lattice L of N sites and only those configurations with an even number of arrows pointing into each vertex are allowed. The six sublattices of L are denoted by A, B, C, D, E, and F, as shown in figure 1. The eight possible configurations allowed at each vertex are shown in figure 2, where each vertex type is assigned a weight. Let the vertex weights be denoted by $\omega_{\alpha,i}$ where $\alpha = \mathrm{A}, \ldots, \mathrm{F}$ and $i = 1, \ldots 8$.

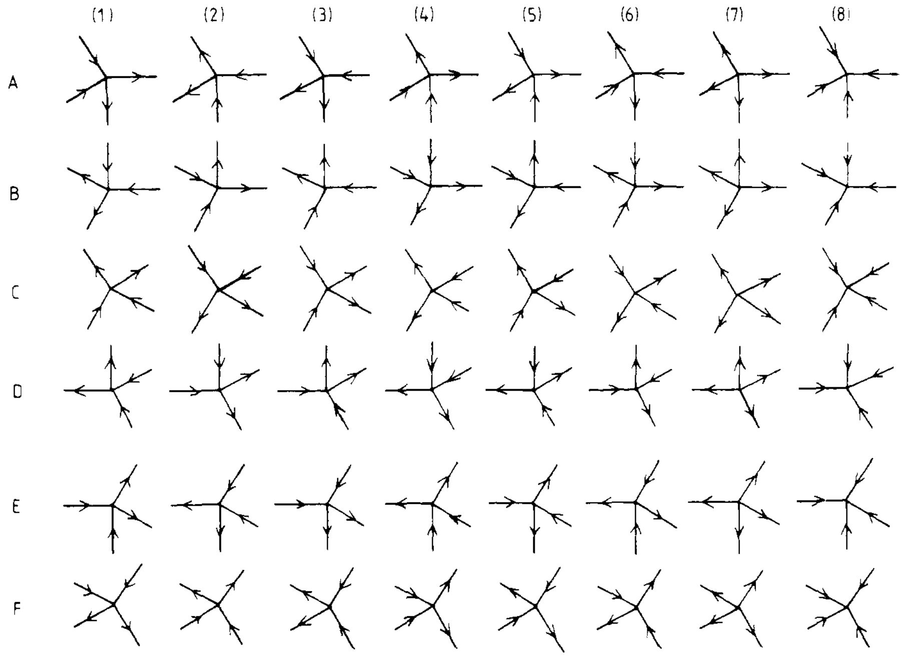

Figure 2. The eight-vertex configurations.

The partition function is

$$Z = \Sigma(\Pi \omega_{\alpha,i}^{n_{\alpha,i}}) \tag{1}$$

where the summation is extended to all allowed arrow configurations on L, and $n_{\alpha,i}$ is the number of the ith-type sites on the α-sublattice. The goal is to compute the free energy

$$\psi = \lim_{N \to \infty} \frac{1}{N} \ln Z. \tag{2}$$

In a physical model, the vertex weights are interpreted as the Boltzmann factors

$$\omega_{\alpha,i} = \exp(-\beta \varepsilon_{\alpha,i}) \tag{3}$$

where $\beta = 1/kT$, k is the Boltzmann constant, T is the temperature, and ε is the vertex energy. The ice model corresponds to the special case $\omega_{\alpha,i} = 0$ for $i = 1, \ldots, 6$ and $\omega_{\alpha,7} = \omega_{\alpha,8} = 0$.

The partition function Z possesses some symmetry relations which follow from general considerations. Reversing all arrows along each triangle implies that Z is invariant under the following exchanges

$$1 \leftrightarrow 5, \qquad 2 \leftrightarrow 6, \qquad 3 \leftrightarrow 8, \qquad 4 \leftrightarrow 7$$

where i denotes $\omega_{\alpha,i}$. Similarly, reversing all the arrows implies that Z is invariant under

$$1 \leftrightarrow 2, \qquad 3 \leftrightarrow 4, \qquad 5 \leftrightarrow 6, \qquad 7 \leftrightarrow 8.$$

Rotational symmetry implies

$$\begin{aligned} Z(\mathrm{A, B, C, D, E, F}) &= Z(\mathrm{C, A, B, F, D, E}) \\ &= Z(\mathrm{D, E, F, A, B, C}) \end{aligned} \tag{4}$$

where A denotes $\omega_{\mathrm{A},i}$, etc.

3. Pfaffian solution

A vertex model can be solved exactly by the method of Pfaffian and dimer city (Kasteleyn 1963) if the free-fermion condition is satisfied at each vertex (Fan and Wu 1970). In our model, the condition reads

$$\omega_{\alpha,1}\omega_{\alpha,2} + \omega_{\alpha,3}\omega_{\alpha,4} = \omega_{\alpha,5}\omega_{\alpha,6} + \omega_{\alpha,7}\omega_{\alpha,8} \tag{5}$$

for all α. Under this condition the partition function is equal to a Pfaffian which is evaluated in the appendix. The result is

$$\psi = \frac{1}{48\pi^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \ln F(\theta, \phi) \tag{6}$$

where

$$\begin{aligned} F = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2 &+ 2(\Omega_2\Omega_3 - \Omega_1\Omega_4)\cos\theta \\ &+ 2(\Omega_2\Omega_4 - \Omega_1\Omega_3)\cos\phi + 2(\Omega_3\Omega_4 - \Omega_1\Omega_2)\cos(\theta-\phi) \\ &- 4a\sin^2(\theta-\phi) - 4b\sin^2\theta - 4c\sin^2\phi \\ &- 4d\sin\theta\sin\phi - 4e\sin\phi\sin(\phi-\theta) - 4f\sin\theta\sin(\theta-\phi) \end{aligned}$$

$$\begin{aligned} \Omega_1 = (111+555)(111+555)^* &+ (222+666)(222+666)^* \\ &+ \Sigma[(436+782)(436+782)^* + (345+871)(345+871)^*] \\ &ijk \equiv \omega_{\mathrm{A},i}\omega_{\mathrm{B},j}\omega_{\mathrm{C},k} \\ &(ijk)^* \equiv \omega_{\mathrm{D},i}\omega_{\mathrm{E},j}\omega_{\mathrm{F},k} \end{aligned}$$

$$\Sigma(ijk)(lmn)^* \equiv (ijk)(lmn)^* + (jki)(mnl)^* + (kij)(nlm)^*$$

$$\begin{aligned} \Omega_2(\mathrm{A, B, C, D, E, F}) = (611+255)&(143+578)^* \\ &+ (166+522)(234+678)^* + (836+382)(764+427)^* \\ &+ (853+318)(471+745)^* + (\mathrm{A} \leftrightarrow \mathrm{D}, \mathrm{B} \leftrightarrow \mathrm{E}, \mathrm{C} \leftrightarrow \mathrm{F}) \\ &\Omega_3 = \Omega_2(\mathrm{C, A, B, F, D, E}) \\ &\Omega_4 = \Omega_2(\mathrm{B, C, A, E, F, D}) \end{aligned}$$

$$
\begin{aligned}
a(\mathrm{A,B,C,D,E,F}) = {} & [(12-78)(3646+2827)+(2523-3768)24+(1668-4823)67] \\
& \times[(12-78)(3646+2827)+(2523-3768)24+(1668-4823)67]^* \\
& +[(12-78)(1718+4535)+(1614-4857)13+(2557-3714)58] \\
& \times[(12-78)(1718+4535)+(1614-4857)13+(2557-3714)58]^* \\
& +\{[(12-78)(1746+4527)+(1614-4857)67+(2557-3714)24] \\
& \times[(4823-1668)13+(3768-2523)58-(12-78)(3618+2835)]^* \\
& +(12-78)(34-78)(34-78)[(56+78)(12-78)(12-78) \\
& +(12-78)(56+78)(56+78)+(1616-4837)16 \\
& +(1648-4825)37+(2525-3748)25+(2537-3716)48]^* \\
& +(\mathrm{A}\leftrightarrow\mathrm{D},\ \mathrm{B}\leftrightarrow\mathrm{E},\ \mathrm{C}\leftrightarrow\mathrm{F})\} \\
& ijjklmn \equiv \omega_{\mathrm{A},i}\omega_{\mathrm{A},j}\omega_{\mathrm{B},k}\omega_{\mathrm{B},l}\omega_{\mathrm{C},m}\omega_{\mathrm{C},n} \\
& (ijklmn)^* \equiv \omega_{\mathrm{D},i}\omega_{\mathrm{D},j}\omega_{\mathrm{E},k}\omega_{\mathrm{E},l}\omega_{\mathrm{F},m}\omega_{\mathrm{F},n} \\
& b = a(\mathrm{C,A,B,F,D,E}) \\
& c = a(\mathrm{B,C,A,E,F,D})
\end{aligned}
$$

$$
\begin{aligned}
d(\mathrm{A,B,C,D,E,F}) = {} & (34-78)(2423-6768)[66(1616-4837)+22(2525-3748) \\
& +26(12-78)(56+78)+26(56+78)(12-78)]^* \\
& +(34-78)(2457-6714)[33(1648-4825)+88(2537-3716) \\
& +38(12-78)(56+78)+38(56+78)(12-78)]^* \\
& +(34-78)(1314-5857)[11(1616-4837)+55(2525-3748) \\
& +15(12-78)(56+78)+15(56+78)(12-78)]^* \\
& +(34-78)(1368-5823)[77(1648-4825)+44(2537-3716) \\
& +47(12-78)(56+78)+47(56+78)(12-78)]^* \\
& +[(1318+5835)(12-78)+18(1316-5837)+35(5825-1348)] \\
& \times[17(4857-1614)+45(3714-2557)-14(12-78)17-57(12-78)45]^* \\
& +[(2435+6718)(12-78)+27(5825-1348)+46(1316-5837)] \\
& \times[17(1668-4823)+45(2523-3768)+14(12-78)36+57(12-78)28]^* \\
& +[(1346+5827)(12-78)+35(2425-6748)+18(6716-2437)] \\
& \times[36(1614-4857)+28(2557-3714)+68(12-78)17+23(12-78)45]^* \\
& +[(2427+6746)(12-78)+27(6748-2425)+46(2437-6716)] \\
& \times[28(3768-2523)+36(4823-1668)-68(12-78)36-23(12-78)28]^* \\
& +(\mathrm{A}\leftrightarrow\mathrm{D},\ \mathrm{B}\leftrightarrow\mathrm{E},\ \mathrm{C}\leftrightarrow\mathrm{F}) \\
& e = d(\mathrm{C,A,B,F,D,E}) \\
& f = d(\mathrm{B,C,A,E,F,D}).
\end{aligned}
$$

The special case of $\omega_{\alpha,7}=\omega_{\alpha,8}=0$ corresponds to the ice-rule vertex model. In this case we have

$$F(\theta, \phi)=|a-b\,e^{i\theta}-b'\,e^{-i\theta}-c\,e^{i\phi}-c'\,e^{-i\phi}-d\,e^{i(\theta-\phi)}-d'\,e^{-i(\theta-\phi)}|^2 \tag{7}$$

where

$$\begin{aligned}
a=(111+555)(111+555)^*+(222+666)(222+666)^*\\
+345(345)^*+453(453)^*+534(534)^*\\
+364(364)^*+643(643)^*+436(436)^*\\
b=314(161+525)^*+(616+252)(423)^*\\
b'=(161+525)(314)^*+423(616+252)^*\\
c=(116+552)(431)^*+342(661+225)^*\\
c'=431(116+552)^*+(661+225)(342)^*\\
d=(611+255)(143)^*+234(166+522)^*\\
d'=143(611+255)^*+(166+522)(234)^*.
\end{aligned}$$

The expression (6) for the free energy is the same as that for the free energy of the eight-vertex free-fermion model on a Kagomé lattice (Lin 1976). It is easy to check that $F(\theta, \phi)=0$ at the following points:

$$\begin{aligned}
&\theta=\phi=0 && \Omega_1=\Omega_2+\Omega_3+\Omega_4\\
&\theta=\pi,\ \phi=0 && \Omega_3=\Omega_1+\Omega_2+\Omega_4\\
&\theta=0,\ \phi=\pi && \Omega_4=\Omega_1+\Omega_2+\Omega_3\\
&\theta=\phi=\pi && \Omega_2=\Omega_1+\Omega_3+\Omega_4.
\end{aligned} \tag{8}$$

In general, all the zeros of $F(\theta, \phi)=0$ are given by (8) and the critical temperature T_c is determined by $\Delta(T_c)=0$ where

$$\Delta(T)=\Omega_1+\Omega_2+\Omega_3+\Omega_4-2\max\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\}. \tag{9}$$

To be specific, we consider the non-analyticity of ψ at $\Omega_1=\Omega_2+\Omega_3+\Omega_4$. Following Hsue *et al* (1975), we expand $F(\theta, \phi)=0$ about $\theta=\phi=0$ and obtain

$$\begin{aligned}
\psi_{\text{singular}} &\sim \int d\theta \int d\phi\, \ln[(\Omega_1-\Omega_2-\Omega_3-\Omega_4)^2+\alpha\theta^2+\beta\theta\phi+\gamma\phi^2]\\
&\sim (T-T_c)^2 \ln|T-T_c|.
\end{aligned} \tag{10}$$

The specific heat diverges logarithmically. The argument breaks down if

$$\beta^2=4\alpha\gamma \tag{11}$$

at T_c (Hsue *et al* 1975). The condition (11) implies that there exist zeros of $F(\theta, \phi)=0$ which are not given by (8). In this case the system exhibits first- or second-order phase transition. The ice-rule vertex model is an example. In this special case the system exhibits second-order phase transition and the specific heat diverges with an exponent $\alpha=\frac{1}{2}$ above the transition temperature (Lin 1975).

Acknowledgment

This research is supported by the National Science Council, Republic of China.

Appendix. Pfaffian solution

Expand each site of a ruby lattice into a 'city' of four terminals to form a dimer lattice. A unit cell of the dimer lattice is shown in figure 3 which corresponds to a 24th-order matrix with elements

$$a(i,j) = -a^*(j,i). \tag{A1}$$

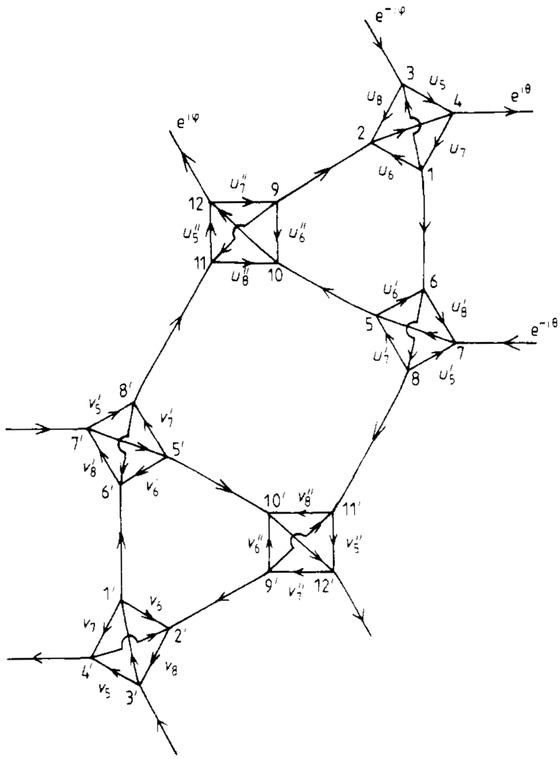

Figure 3. A unit cell of the dimer lattice which corresponds to a 24th-order matrix.

A periodic boundary condition is assumed. The sign of each element is identified by an arrow such that its pointing from i to j implies sgn $a(i,j) = +1$. A polygon with an odd number of clockwise sides is called clockwise odd. Arrows are arranged so that every closed polygon is clockwise odd. Following the same procedure as Hsue *et al* (1975), we have

$$\psi = \frac{1}{48\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln[(\omega_{A,2}\omega_{B,2}\omega_{C,2}\omega_{D,2}\omega_{E,2}\omega_{F,2})^2 D(\theta,\phi)] \tag{A2}$$

where $D(\theta, \phi)$ is the determinant of the matrix $a(i,j)$ and the non-vanishing matrix elements associated with positive signs are:

$$a(1,3)=u_3,\qquad a(2,4)=u_4,\qquad a(3,4)=u_5,\qquad a(1,2)=u_6,$$
$$a(4,1)=u_7,\qquad a(3,2)=u_8$$
$$a(7,5)=u_3',\qquad a(6,8)=u_4',\qquad a(8,7)=u_5',\qquad a(5,6)-u_6',$$
$$a(8,5)=u_7',\qquad a(6,7)=u_8'$$
$$a(9,11)=u_3'',\qquad a(10,12)=u_4'',\qquad a(11,12)=u_5'',\qquad a(9,10)=u_6'',$$
$$a(12,9)=u_7'',\qquad a(11,10)=u_8''$$
$$a(3',1')=v_3,\qquad a(4',2')=v_4,\qquad a(3',4')=v_5,\qquad a(1',2')=v_6,$$
$$a(1',4')=v_7,\qquad a(2',3')=v_8$$
$$a(7',5')=v_3',\qquad a(8',6')=v_4',\qquad a(7',8')=v_5',\qquad a(5',6')=v_6',$$
$$a(5',8')=v_7',\qquad a(6',7')=v_8'$$
$$a(9',11')=v_3'',\qquad a(10',12')=v_4'',\qquad a(11',12')=v_5'',\qquad a(9',10')=v_6'',$$
$$a(12',9')=v_7'',\qquad a(11',10')=v_8''$$
$$a(1,6)=a(1',6')=a(9,2)=a(9',2')=a(5,10)=a(5',10')=a(8,11')$$
$$=a(8',11)=1$$
$$a(4,7')=\exp(\mathrm{i}\theta),\qquad a(4',7)=\exp(-\mathrm{i}\theta),\qquad a(12,3')=\exp(\mathrm{i}\phi),$$
$$a(12',3)=\exp(-\mathrm{i}\phi)$$
$$u_i\equiv\omega_{A,i}/\omega_{A,2},\qquad u_i'\equiv\omega_{B,i}/\omega_{B,2},\qquad u_i''\equiv\omega_{C,i}/\omega_{C,2}$$
$$v_i\equiv\omega_{D,i}/\omega_{D,2},\qquad v_i'\equiv\omega_{E,i}/\omega_{E,2},\qquad v_i''\equiv\omega_{F,i}/\omega_{F,2}.$$

Equation (A2) reduces to equation (6) after some algebra.

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